

Lecture 17

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8.3 - Moments and Centers of Mass

1 dimension (discrete case)

(This case does not even require calculus.)

Suppose we have a system of point masses m_1, m_2, \dots, m_n which are placed on the x -axis at points x_1, x_2, \dots, x_n . Then, the moment of the system about the origin is

$$M_0 =$$

and the center of mass of the system is

$$\bar{x} =$$

If you imagine the x -axis above as a rod of negligible mass, then the center of mass is the point on the rod where you could balance the rod on your fingertip. The moment measures the rod's tendency to rotate if you try to balance it at the origin: clockwise if $M_0 > 0$, counterclockwise if $M_0 < 0$, balanced if $M_0 = 0$.

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If we replace the system with one where all of the mass is concentrated at the center of mass, then it will have the same moment as the original system.

2 dimensions (discrete case)

This time, our masses m_1, \dots, m_n are distributed around the xy -plane at points $(x_1, y_1), \dots, (x_n, y_n)$. This time, we have moments about the x - & y -axes:

Moment about y -axis: $M_y =$

Moment about x -axis: $M_x =$

The center of mass of this system is the point (\bar{x}, \bar{y}) in the plane given by:

$$\bar{x} = \quad , \quad \bar{y} =$$

Of course, not many systems are discrete, Calculus allows us to deal with this case:

2 dimensions (continuous)

Suppose we have a region R in the plane with uniform (constant) density (we need double integrals to deal with non-uniform density, cf., calc III); we can find the mass, moments about the x -& y -axes, and center of mass of this object too.

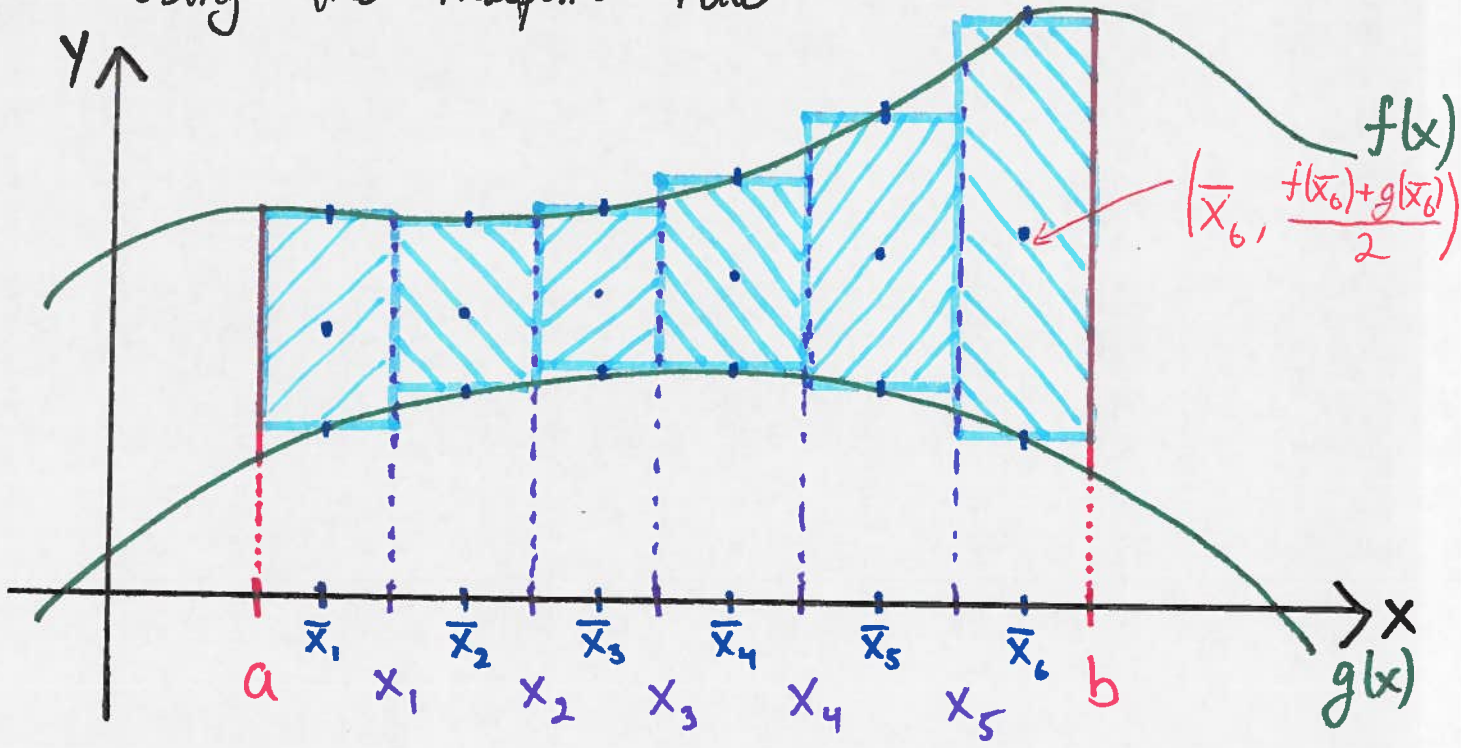
Let's suppose R is a region between two functions $y=f(x)$ & $y=g(x)$, i.e.,

$$R = \{(x,y) \mid a \leq x \leq b, g(x) \leq y \leq f(x)\},$$

and that R has constant density ρ . We compute the mass and moments using Riemann sums to find the integrals. Riemann sums use rectangles, so first we need to know where the center of mass of a rectangle with constant density is. It isn't too hard to believe that the center of mass of a rectangle $[a,b] \times [c,d]$ is its "center" $(\frac{a+b}{2}, \frac{c+d}{2})$.



Since the center of mass is in the middle of the rectangle, we should set up the Riemann sums using the midpoint rule:



First, the mass:

$m =$ _____

The moments about the x- & y- axes:

Of course, the center of mass, as before,
is given by

$$(\bar{x}, \bar{y}) =$$

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Ex: Find the centroid (another name for center of mass)
of the region bounded by $y = 1 - x^2$ and the x-axis.

Ex: Find the mass and center of mass of the region between the curves $y=x^2$ and $x=y^2$ if it has constant density $\rho=6$.